Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MSO202A/MSO202 Quiz 2 (September 7, 2013) Grading Scheme
Introduction To Complex Analysis

Roll No.: ………………… Section: …………. **Time: 40 Minutes**
Marks: 40

Name: ……………………………………………….

Note: Give only answers (no details of workout) for Problems 1 to 8 at dotted lines. Each of these problems is of 5 marks.

1. Which of the sets $S_1 = \{z : \text{Im}\{\frac{z-2i}{2}\} > 0\}$, $S_2 = \{z : \text{Im}\{\frac{z-2i}{4}\} < 0\}$, $S_3 = \{z : \text{Im}\{\frac{z-1-2i}{4}\} > 0\}$, $S_4 = \{z : \text{Im}\{\frac{z-1-2i}{2}\} < 0\}$ represent the half-plane $H = \{z : |z-i| < |z-3i|\}$:

   (i) …………………………… (ii) …………………………

Solution: Observe that H is the half-plane below the line $y = 2$, which is also the left-half plane passing through either of the points $2i$ or $1+2i$, with the direction of their boundary line determined by the unit vector $-\hat{i}$. Therefore, the correct answers are S_2 and S_4.
(3 marks for one correct answer, 5 marks for both correct answers, deduct 3 marks for one extra answer and all 5 marks for two extra answers)

2. All the values of z for which $\sin z = \cosh 3$ are given by

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Solution: $\sin z = \cosh 3 \Rightarrow \sin x \cosh y + i \cos x \sinh y = \cosh 3$
$\Rightarrow \cos x \sinh y = 0$ *and* $\sin x \cosh y = \cosh 3$
$\Rightarrow x = 2n\pi + \frac{\pi}{2}$ *and* $y = \pm 3 \Rightarrow z = (2n\pi + \frac{\pi}{2}) \pm i\, 3$, *where n is an integer.*
(5 marks, deduct 1 mark if values of z with correct values of x and $y = 0$ are also included)

3. The value of the integral $\oint_{|z|=2} Log\, z\, dz$, where the circle $|z| = 2$ is oriented clockwise, is

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Solution: $\int_{|z|=2} Log\, z\, dz = -\int_{-\pi}^{\pi} [\ln 2 + i\theta].2e^{i\theta}.i\, d\theta = 2\int_{-\pi}^{\pi} \theta e^{i\theta}\, d\theta = 4\pi i.$ ***(5 marks)***

4. The order of zero of the function $h(z)=4\cos z^4+2z^8-4$ at $z=0$ is

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Solution: The Taylor's expansion of $h(z)=\cos z^4$ *around the point* $z=0$ *is*

$$\cos z^4 = 1-\frac{z^8}{2!}+\frac{z^{16}}{4!}+\ldots+(-1)^n\frac{z^{8n}}{(2n)!}+\ldots$$

Therefore, the first nonzero term in Taylor's expansion of $h(z)=4\cos z^4+2z^8-4$ *around* $z=0$ *is* $\frac{z^{16}}{3!}$.

$\Rightarrow$ *Order of zero of* $h(z)$ *at* $z=0$ *is* 16. ***(5 marks)***

5. The largest domain in which the functions $f_1(z)=\frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n$ and $f_2(z)=\sum_{n=0}^{\infty}z^n$ are equal, is

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Solution: Both the given functions are equal to $\frac{1}{1-z}$ *in the domain* $\{z:|z-i|<\sqrt{2}\}\cap\{z:|z|<1\}$ *which is the largest such domain.* ***(5 marks)***

6. On which of the sets $T_1=\{\frac{1+i}{2n^{1/n}}\}_{n=1}^{\infty}, T_2=\{\frac{1+i}{(2n)^{1/n}}\}_{n=1}^{\infty}, T_3=\{\frac{1-i}{(2n)^{1/n}}\}_{n=1}^{\infty}, T_4=\{\frac{1-i}{2n^{1/n}}\}_{n=1}^{\infty}$, a function $f(z)$ (not identically zero) analytic in the square region $D=\{z=x+iy:-1<x<1,\ -1<y<1\}$ can have its zeros:

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Solution: The sequences of points in the sets T_2 *and* T_3 *converge to* $1+i$ *and* $1-i$ *respectively, which do not lie in the region of analyticity of* $f(z)$, *while sequences of points in the sets* T_1 *and* T_4 *converge to* $\frac{1+i}{2}$ *and* $\frac{1-i}{2}$ *respectively, lying in the region of analyticity of* $f(z)$. *Therefore, by Isolated Zeros Theorem,* $f(z)$ *can have zeros only in the sets* T_2 *and* T_3.

(3 marks for one correct answer, 5 marks for both correct answers, deduct 3 marks for one extra answer and all 5 marks for two extra answers)

7. Let *f(z)* be analytic in the whole complex plane ***C*** and $|f(z)| \geq 1$ for all $z \in$ ***C***. If $f(0)=1$, then $f(1)=$

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Solution: $|f(z)| \geq 1$ *for all* $z \in$ ***C*** $\Rightarrow$ *f(z)* $\neq$ *0 in* ***C*** $\Rightarrow \dfrac{1}{f(z)}$ *is an entire function*

Again, $|f(z)| \geq 1$ *for all* $z \in$ ***C*** $\Rightarrow \left|\dfrac{1}{f(z)}\right| \leq 1$ *for all* $z \in$ ***C****. Thus,* $\dfrac{1}{f(z)}$ *is entire and bounded in* ***C***

$\Rightarrow$ *(By Liouville theorem)* $\dfrac{1}{f(z)}$ *is a constant function.* $\Rightarrow$ $f(z)$ *is a constant function in* ***C*** $\Rightarrow f(1)=1$.

(5 marks)

8. Let C be the circle $|z-2\pi|=2$ described once in clockwise direction. Then, the value of $Q=\oint_C \dfrac{\cos z}{(z-\pi)(z-2\pi)^3}dz$, is

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Solution: By Cauchy Integral Formula for derivatives,

$$Q=-2\pi i\times\frac{1}{2!}\left[\frac{d^2}{dz^2}[\frac{\cos z}{z-\pi}]\right]_{z=2\pi} = -\pi i\times\left[\frac{d}{dz}(\frac{-(\sin z)(z-\pi)-\cos z}{(z-\pi)^2})\right]_{z=2\pi} \quad \text{……………………}(*)$$

$$=-\pi i\times\left[\frac{[-(\cos z)(z-\pi)-\sin z]\times(z-\pi)^2-[-(\sin z)(z-\pi)-\cos z]\times 2(z-\pi)}{(z-\pi)^4}\right]_{z=2\pi}$$

$$=-\pi i\times\frac{-\pi\times\pi^2+2\pi}{\pi^4}=i(\frac{\pi^2-2}{\pi^2}).$$

(5 marks, deduct one mark if given answer is negative of right answer, deduct one mark if answer is left at step (*))